

Linear energy divergences in Coulomb gauge QCD

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Abstract

The structure of linear energy divergences is analysed on the example of one graph to 3-loop order. Such dangerous divergences do cancel when all graphs are added, but next to leading divergences do not cancel out.

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1 Introduction

The Coulomb gauge in non-Abelian gauge theories is a very good example of a physical gauge. It is manifestly unitary. Although there are ghosts, their propagators have no poles. The propagators are closely related to the polarization states of real spin-1 particles. Nevertheless there are problems concerned with energy divergences [1]. In individual Feynman graphs there appear even linear energy divergences. These are divergences over the energy integration in a loop, for fixed values of the 3-momentum, of the form

$$\int dk_0 F \quad (1)$$

where F is independent of k_0 . They do cancel when all graphs are combined [2]. However, it makes one uneasy in manipulating divergent and unregulated integrals.

2 The graph $2B(b, 0i0)$

We have studied the renormalization in Coulomb gauge QCD to three-loop order in Hamiltonian formalism [3]. It was shown that to three loops the UV divergences cannot be consistently absorbed by the Christ-Lee term [4]. In this paper we show in detail how dangerous the linear divergences are on the example of one graph with fermion loop three-point function. The graph is shown in fig.1.

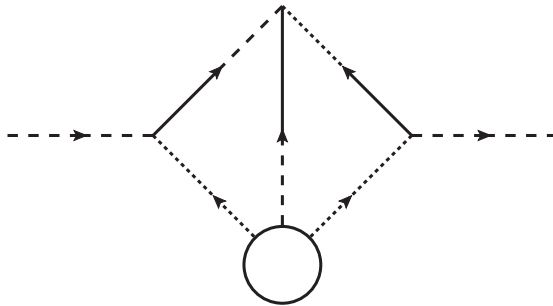


Figure 1: Graph $2B(b, 0i0)$ which is an example of the graph containing linear energy divergences

We use the same notation and graphical conventions as in [5]. Using the Ward identities for high energies we have derived the expression for the quark loop three-point function with two Coulomb and one transverse line,

$$V_{00i}(k_1, k_2, k_3) \approx \frac{K_{2i}}{k_{10}} [k_{20}S(k_2) + k_{30}S(k_3)] - \frac{K_{1i}}{k_{20}} [k_{30}S(k_3) + k_{10}S(k_1)] , \quad (2)$$

where the gluon self-energy from the quark loop is

$$\text{tr}(t^a t^b) S_{\mu_1 \mu_2}(p) = g^2 C_q \delta^{ab} (p_{\mu_1} p_{\mu_2} - p^2 \delta_{\mu_1 \mu_2}) S(p^2) \quad (3)$$

with

$$S(p^2) = 8i\pi^{2-\frac{\epsilon}{2}} \Gamma(\frac{\epsilon}{2}) \frac{\Gamma^2(2-\frac{\epsilon}{2})}{\Gamma(4-\epsilon)} [(-p^2 - i\eta)^{-\frac{\epsilon}{2}} - (\mu^2)^{-\frac{\epsilon}{2}}], \quad (4)$$

where a renormalization subtraction at a mass μ has been made and $\epsilon = 4-n$, n is the number of space-time dimensions. Applying (2), (3) and (4) to the graph in fig.1 we obtain for the K^2 part the expression

$$\begin{aligned} 2B(b, 0i0) = & -\frac{1}{4} g^6 (2\pi)^{-8} C_q^2 T(R) \delta_{ab} K^2 \int d^4 p \int d^4 q \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \\ & \times \left\{ \frac{1}{p_0 p'_0} [S(r') - S(q)] + \frac{1}{p_0 q_0} [S(r') - S(p')] - \frac{1}{p_0 r'_0} [S(q) + S(p')] \right\}. \end{aligned} \quad (5)$$

The momenta are defined as $p' = p - k$, $q' = q - k$, $r' = k - p - q$, $p^2 = p_0^2 - P^2$ and in the high energy limit we have used the approximation

$$\frac{p_0}{p_0^2 - P^2 + i\eta} \approx \frac{1}{p_0}. \quad (6)$$

The first term in (5) is explicit linear energy divergence. It is the difference of two integrals, one with $S(r')$ and the other with $S(q)$.

3 Linear energy divergence

Let us consider the first integral in (5).

$$\begin{aligned} J_{ij} = & \Gamma(\frac{\epsilon}{2}) \int d^4 p \int d^4 q \frac{p_0}{p_0^2 - P^2 + i\eta} \cdot \frac{p'_0}{p_0'^2 - P'^2 + i\eta} \\ & \times \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} [(p_0 + q_0 - k_0)^2 - (P + Q - K)^2 + i\eta]^{-\frac{\epsilon}{2}} \end{aligned} \quad (7)$$

Using the Schwinger representation for the propagators J_{ij} becomes

$$\begin{aligned} J_{ij} = & (-i)^{2+\frac{\epsilon}{2}} \int_{-\infty}^{\infty} dp_0 p_0 (p - k)_0 \int d^{3-\epsilon} P \int_{-\infty}^{\infty} dq_0 \int d^{3-\epsilon} Q \\ & \times \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \int_0^{\infty} d\gamma \gamma^{\frac{\epsilon}{2}-1} \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \\ & \times e^{i\alpha(p_0^2 - P^2 + i\eta)} e^{i\beta(p_0'^2 - P'^2 + i\eta)} e^{i\gamma(r_0'^2 - R'^2 + i\eta)} \end{aligned} \quad (8)$$

Performing the q_0 and p_0 integrations with Gaussian integrals followed by integration over the parameter γ , we obtain

$$J_{ij} = (-i)^{3+\frac{\epsilon}{2}} \pi \Gamma\left(\frac{\epsilon-1}{2}\right) \int d^{3-\epsilon} P \int d^{3-\epsilon} Q \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-i\alpha P^2 - i\beta P'^2 - \eta(\alpha+\beta)} \\ \times \frac{1}{(\alpha+\beta)^{3/2}} e^{ik_0^2 \frac{\alpha\beta}{\alpha+\beta}} \cdot \left[\frac{i}{2} - \frac{\alpha\beta}{\alpha+\beta} k_0^2 \right] \cdot (\eta + iR'^2)^{\frac{1-\epsilon}{2}} \quad (9)$$

Changing the variables of integration α and β as

$$\alpha = \lambda v, \quad \beta = \lambda(1-v), \quad \left(\frac{\partial\alpha, \partial\beta}{\partial\lambda, \partial v} \right) = \lambda, \\ 0 < v < 1, \quad 0 < \lambda < \infty, \quad (10)$$

makes λ -integration easy, leading to

$$J_{ij} = \Gamma\left(\frac{\epsilon-1}{2}\right) \frac{1}{2} \pi^{\frac{3}{2}} e^{-i\epsilon\frac{\pi}{2}} \int d^{3-\epsilon} P \int d^{3-\epsilon} Q \int_0^1 dv \frac{P^2 v + P'^2(1-v)}{[P^2 v + P'^2(1-v) - k_0^2 v(1-v) - i\eta]^{3/2}} \\ \times \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \cdot \frac{1}{(R'^2)^{\frac{\epsilon-1}{2}}}. \quad (11)$$

Repeating the same operations with the other integral containing $S(q)$, we obtain for the linear energy divergence term in (5) the expression

$$L_{ij} = \int d^4 p \int d^4 q \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \cdot \frac{1}{p_0 p'_0} [S(r') - S(q)] \\ = \Gamma\left(\frac{\epsilon-1}{2}\right) \frac{1}{2} \pi^{3/2} e^{-i\epsilon\frac{\pi}{2}} \int d^{3-\epsilon} P \int d^{3-\epsilon} Q \int_0^1 dv \frac{P^2 v + P'^2(1-v)}{[P^2 v + P'^2(1-v) - k_0^2 v(1-v) - i\eta]^{3/2}} \\ \times \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \left[\frac{1}{(R'^2)^{\frac{\epsilon-1}{2}}} - \frac{1}{(Q'^2)^{\frac{\epsilon-1}{2}}} \right]. \quad (12)$$

The linear energy divergence reflects in the factor $\Gamma(\frac{\epsilon-1}{2})$. The Q - integral is

$$X_j = \int d^{3-\epsilon} Q \frac{Q'_j}{Q^2 Q'^2} \left[\frac{1}{(R'^2)^{\frac{\epsilon-1}{2}}} - \frac{1}{(Q'^2)^{\frac{\epsilon-1}{2}}} \right]. \quad (13)$$

The second integral is easy.

$$B_j = \int d^{3-\epsilon} Q \frac{Q'_j}{Q^2 (Q'^2)^{\frac{1+\epsilon}{2}}} \\ = -K_j (K^2)^{-\epsilon} \pi^{\frac{3-\epsilon}{2}} \frac{\Gamma(\epsilon)}{\Gamma(\frac{1+\epsilon}{2})} \cdot \frac{\Gamma(2-\epsilon) \Gamma(\frac{1-\epsilon}{2})}{\Gamma(\frac{5}{2} - \frac{3\epsilon}{2})} \quad (14)$$

Let us study the first integral in (13).

$$A_j = \int d^{3-\epsilon} Q \frac{Q'_j}{Q^2 Q'^2} \cdot \frac{1}{(R'^2)^{\frac{\epsilon-1}{2}}} \quad (15)$$

We combine the denominators Q^2 and Q'^2 with the Feynman parameter x and then $(R'^2)^{\frac{\epsilon-1}{2}}$ with the parameter y (remembering that $R' = -P' - Q$, $Q' = Q - K$), then integrate in $d^{3-\epsilon} Q$.

$$A_j = \pi^{\frac{3-\epsilon}{2}} \frac{\Gamma(\epsilon)}{\Gamma(\frac{\epsilon-1}{2})} \int_0^1 dx \int_0^1 dy y^{\frac{\epsilon-3}{2}} (1-y) \cdot [K_j x(1-y) - K_j - P'_j y] \times \{P'^2 y + K^2 x(1-y) - [Kx(1-y) - P'y]^2\}^{-\epsilon} \quad (16)$$

We insert (16) and (14) into (12).

$$L_{ij} = \frac{1}{2} \pi^{3-\frac{\epsilon}{2}} e^{-i\epsilon \frac{\pi}{2}} \int d^{3-\epsilon} P \int_0^1 dv \frac{P^2 v + P'^2 (1-v)}{[P^2 v + P'^2 (1-v) - k_0^2 v(1-v) - i\eta]^{\frac{3}{2}}} \cdot \frac{P_i}{P^2 P'^2} \times \{\Gamma(\epsilon) \int_0^1 dx \int_0^1 dy y^{\frac{\epsilon-3}{2}} (1-y) [K_j x(1-y) - K_j - P'_j y] \cdot [P'^2 y + K^2 x(1-y) - (Kx(1-y) - P'y)^2]^{-\epsilon} + \Gamma(\frac{\epsilon-1}{2}) \Gamma(\epsilon) \cdot \frac{\Gamma(2-\epsilon) \Gamma(\frac{1-\epsilon}{2})}{\Gamma(\frac{1+\epsilon}{2}) \Gamma(\frac{5}{2} - \frac{3\epsilon}{2})} \cdot K_j (K^2)^{-\epsilon}\} \quad (17)$$

We notice that the $d^{3-\epsilon} P$ integration is IR safe and also UV finite by power counting for K_j terms, while it is UV divergent for $P'_j y$ term. Hence, for the leading divergence we can set

$$[P'^2 y + K^2 x(1-y) - (Kx(1-y) - P'y)^2]^{-\epsilon} \approx 1, \quad (18)$$

$$L_{ij} = \Gamma(\frac{\epsilon-1}{2}) \Gamma(\epsilon) \frac{1}{2} \pi^3 K_j \int d^{3-\epsilon} P \int_0^1 dv \frac{P_i}{P^2 P'^2} \cdot \frac{P^2 v + P'^2 (1-v)}{[P^2 v + P'^2 (1-v) - k_0^2 v(1-v) - i\eta]^{\frac{3}{2}}} \times \left\{ \frac{\Gamma(3)}{2\Gamma(\frac{\epsilon+5}{2})} - \frac{\Gamma(2)}{\Gamma(\frac{\epsilon+3}{2})} + \frac{\Gamma(2-\epsilon) \Gamma(\frac{1-\epsilon}{2})}{\Gamma(\frac{1+\epsilon}{2}) \Gamma(\frac{5}{2} - \frac{3\epsilon}{2})} \right\} - \frac{1}{2} \pi^3 \Gamma(\epsilon) \int d^{3-\epsilon} P \int_0^1 dv \frac{P^2 v + P'^2 (1-v)}{[P^2 v + P'^2 (1-v) - k_0^2 v(1-v) - i\eta]^{3/2}} \cdot \frac{P_i P'_j}{P^2 P'^2} \int_0^1 dx \int_0^1 dy y^{\frac{\epsilon-1}{2}} (1-y). \quad (19)$$

We can easily isolate the UV divergence from the last integral in (19). It behaves like $\Gamma(\frac{\epsilon}{2})$.

4 Conclusion

Individual Feynman graphs in Coulomb gauge QCD to three loop order contain even linear energy divergences. Our analysis shows their behaviour. It is

$$L_{ij} = \Gamma(\frac{\epsilon-1}{2})\Gamma(\epsilon)K_iK_jf(k_0^2, K^2) + \Gamma(\epsilon)\Gamma(\frac{\epsilon}{2})K_iK_jg(k_0^2, K^2). \quad (20)$$

The first term is the product of poles in $\frac{1}{\epsilon-1} \cdot \frac{1}{\epsilon}$ while the second is a double pole in $\frac{1}{\epsilon^2}$. Such dangerous divergences do cancel in the sum of graphs with three-point and four-point fermion loop insertions, but next to leading divergences coming from $\frac{1}{p_0q_0}$, $\frac{1}{p_0r'_0}$ and $\frac{1}{q_0r'_0}$ terms do not. Hence, UV divergences from higher order graphs cannot be consistently absorbed by renormalization of the Christ-Lee term [3].

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